

Geometry I, 2006 : Coursework 9 Solutions

2. (i) [2 marks] A function $t : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *linear transformation* if for all $u, v \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ we have $t(u + v) = t(u) + t(v)$ and $t(\alpha u) = \alpha t(u)$.

(ii) [2 marks each; 1 mark if answer correct, but justification only partly correct; no marks if wrong answer or no justification or wrong justification.]

Let $t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a + 2b \end{pmatrix}$. Then t is a linear transformation.

This can be justified directly by checking that t satisfies the definition given in (i), or more simply by noting that, for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, we have $t \begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The function $t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 \\ b \end{pmatrix}$ is not a linear transformation, since, for example, $t \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

3. [1 mark] The matrix representing s is $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 3 \end{pmatrix}$.

[1 mark] The matrix representing t is $B = \begin{pmatrix} -3 & 1 & 2 \\ 4 & -2 & 1 \end{pmatrix}$.

[2 marks] The matrix representing $s \circ t$ is $AB = \begin{pmatrix} -7 & 3 & 1 \\ -1 & -1 & 8 \\ 0 & -2 & 11 \end{pmatrix}$.

4. [3 marks] Let $u \in \mathbb{R}^n$. Then

$$(s + t)(u) = s(u) + t(u) \quad (\text{by definition of } s + t)$$

$$= Au + Bu \quad (\text{since } A, B \text{ represent } s, t)$$

$$= (A + B)u \quad (\text{due to the right distributivity of matrix multiplication over matrix addition}).$$

5. [3 marks] Let $u \in \mathbb{R}^n$. Then

$$(s \circ t)(u) = (AB)u \quad (\text{shown in lectures})$$

$$= I_n u \quad (\text{since } B = A^{-1})$$

$$= u.$$

6. (i) [2 marks]

$$\begin{aligned} S_\theta S_\theta &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} (\cos 2\theta)^2 + (\sin 2\theta)^2 & 0 \\ 0 & (\sin 2\theta)^2 + (\cos 2\theta)^2 \end{pmatrix} = I_2. \end{aligned}$$

$$\text{Thus } (S_\theta)^{-1} = S_\theta.$$

(ii) [2 marks] [There are (at least) two ways of doing this. Both these ways use part (i) that shows that a reflection is its own inverse. The first proof also uses the fact (proved in lectures) that if A and B are invertible $n \times n$ matrices then $(AB)^{-1} = B^{-1}A^{-1}$. The second proof checks directly that the required properties to be an inverse hold.]

First proof:

$$(S_{\theta_1}S_{\theta_2}S_{\theta_3})^{-1} = (S_{\theta_3})^{-1}(S_{\theta_1}S_{\theta_2})^{-1} = (S_{\theta_3})^{-1}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} = S_{\theta_3}S_{\theta_2}S_{\theta_1}.$$

Second proof:

$$\begin{aligned} (S_{\theta_1}S_{\theta_2}S_{\theta_3})(S_{\theta_3}S_{\theta_2}S_{\theta_1}) &= S_{\theta_1}S_{\theta_2}S_{\theta_3}(S_{\theta_3})^{-1}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} = S_{\theta_1}S_{\theta_2}I_2(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} \\ &= S_{\theta_1}S_{\theta_2}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} = S_{\theta_1}I_2(S_{\theta_1})^{-1} = S_{\theta_1}(S_{\theta_1})^{-1} = I_2. \end{aligned}$$

Similarly, $(S_{\theta_3}S_{\theta_2}S_{\theta_1})(S_{\theta_1}S_{\theta_2}S_{\theta_3}) = I_2$.

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